

# A CRITERIA FOR CORRECT SOLVABILITY IN $L_p(\mathbb{R})$ OF A GENERAL STURM-LIOUVILLE EQUATION

N.A. CHERNYAVSKAYA AND L.A. SHUSTER

ABSTRACT. We consider an equation

$$-(r(x)y'(x))' + q(x)y(x) = f(x), \quad x \in \mathbb{R} \quad (1)$$

where  $f \in L_p(\mathbb{R})$ ,  $p \in (1, \infty)$  and

$$r > 0, \quad q \geq 0, \quad \frac{1}{r} \in L_1^{\text{loc}}(\mathbb{R}), \quad q \in L_1^{\text{loc}}(\mathbb{R}), \quad (2)$$

$$\int_{-\infty}^0 \frac{dt}{r(t)} = \int_0^{\infty} \frac{dt}{r(t)} = \infty. \quad (3)$$

By a solution of (1) we mean any function  $y$  which is absolutely continuous together with  $ry'$  and satisfies (1) almost everywhere on  $\mathbb{R}$ . Under conditions (2)–(3), we give a criterion for correct solvability of (1) in  $L_p(R)$ ,  $p \in (1, \infty)$ .

## 1. INTRODUCTION

In the present paper, we consider an equation

$$-(r(x)y'(x))' + q(x)y(x) = f(x), \quad x \in \mathbb{R} \quad (1.1)$$

where  $f \in L_p(\mathbb{R})$ ,  $(L_p(\mathbb{R}) := L_p)$ ,  $p \in (1, \infty)$  and

$$r > 0, \quad q \geq 0, \quad \frac{1}{r} \in L_1^{\text{loc}}(\mathbb{R}), \quad q \in L_1^{\text{loc}}(\mathbb{R}). \quad (1.2)$$

In the sequel, by a solution of equation (1.1), we mean any function  $y$  which is absolutely continuous together with  $ry'$  and satisfies (1.1) almost everywhere on  $\mathbb{R}$ . We say that equation (1.1) is correctly solvable in a given space  $L_p$ ,  $p \in [1, \infty)$  if the following assertions I)–II) hold (see [7, Ch.III, §6, no.2]):

- 1) for any function  $f \in L_p$ , there exists a unique solution of (1.1),  $y \in L_p$ ;
- 2) there exists an absolute constant  $c(p) \in (0, \infty)$  such that the solution of (1.1),  $y \in L_p$ , satisfies the inequality

$$\|y\|_p \leq c(p)\|f\|_p, \quad \forall f \in L_p \quad (\|f\|_p := \|f\|_{L_p}). \quad (1.3)$$

Our goal is to find exact requirements of  $r$  and  $q$  which, for a given  $p \in (1, \infty)$ , guarantee correct solvability of (1.1) in  $L_p$ . In the sequel, for brevity, this is referred to as “problem

I)–II)” or “question on I)–II)”. Note that problem I)–II) can be reformulated in other terms (see [1, 13]).

To this end, let us introduce the set  $\mathcal{D}_p$  and the operator  $\mathcal{L}_p$  :

$$\mathcal{D}_p = \{y \in L_p : y, ry' \in AC^{\text{loc}}(\mathbb{R}), \quad -(ry')' + qy \in L_p\},$$

$$\mathcal{L}_p y = -(ry')' + qy, \quad y \in \mathcal{D}_p.$$

The linear operator  $\mathcal{L}_p$  is called the maximal Sturm-Liouville operator, and problem I)–II) is, evidently, equivalent to the problem of existence and boundedness of the operator  $\mathcal{L}_p^{-1} : L_p \rightarrow L_p$ , i.e., to the problem of continuous invertibility of the operator  $\mathcal{L}_p$  (see [1]). The question on I)–II) in the first or second formulation was studied in [18, 19, 5] for  $r \equiv 1$  and in [13, 3] for  $r \not\equiv 1$ . Finally, note that the problem of continuous invertibility of the minimal Sturm-Liouville operator  $\mathcal{L}_{o,p}$  was considered in [13, 1, 14]. This operator is defined as the closure in  $L_p$  of the operator  $\mathcal{L}'_{o,p}$  :

$$\mathcal{L}'_{o,p} = -(ry')' + qy, \quad y \in \mathcal{D}'_{o,p}$$

where the set  $\mathcal{D}'_{o,p}$  consists of all finitary functions belonging to  $\mathcal{D}_p$ . The operator  $\mathcal{L}_{o,p}$  was studied in [13, 1] for  $p \in [1, \infty)$ , and in [14] for  $p = 2$ . See [1] for a brief survey of the work on continuous invertibility of the Sturm-Liouville operators of both types.

Let us now return to the initial question on I)–II) and present our results.

**Theorem 1.1.** *Suppose that assumptions (1.2) hold and, in addition,*

$$\int_{-\infty}^0 \frac{dt}{r(t)} = \int_0^{\infty} \frac{dt}{r(t)} = \infty. \quad (1.4)$$

*Then if equation (1.1) is correctly solvable in  $L_p$ ,  $p \in [1, \infty)$ , the following conditions hold:*

$$\int_{-\infty}^x q(t)dt > 0, \quad \int_x^{\infty} q(t)dt > 0 \quad \forall x \in \mathbb{R}, \quad (1.5)$$

$$\lim_{|d| \rightarrow \infty} \int_{x-d}^x \frac{dt}{r(t)} \cdot \int_{x-d}^x q(t)dt = \infty \quad \forall x \in \mathbb{R}. \quad (1.6)$$

*Remark 1.2.* Conditions (1.5) and (1.6) were introduced in [5] and [3].

Below we need the following lemma.

**Lemma 1.3.** [3] *Under conditions (1.2) and (1.5), the equation*

$$(r(x)z'(x))' = q(x)z(x), \quad x \in \mathbb{R} \quad (1.7)$$

*has a fundamental system of solutions (FSS)  $\{u, v\}$  with the following properties:*

$$v(x) > 0, \quad u(x) > 0, \quad v'(x) \geq 0, \quad u'(x) \leq 0 \quad \forall x \in \mathbb{R}, \quad (1.8)$$

$$r(x)[v'(x)u(x) - u'(x)v(x)] = 1, \quad \forall x \in \mathbb{R}, \quad (1.9)$$

$$\lim_{x \rightarrow -\infty} \frac{v(x)}{u(x)} = \lim_{x \rightarrow +\infty} \frac{u(x)}{v(x)} = 0, \quad (1.10)$$

$$\int_{-\infty}^0 \frac{dt}{r(t)v^2(t)} = \int_0^{\infty} \frac{dt}{r(t)u^2(t)} = \infty, \quad \int_0^{\infty} \frac{dt}{r(t)v^2(t)} < \infty, \quad \int_{-\infty}^0 \frac{dt}{r(t)u^2(t)} < \infty. \quad (1.11)$$

Moreover, properties (1.8)–(1.11) determine the FSS  $\{u, v\}$  uniquely up to positive constant factors inverse one to another.

The FSS from Lemma 1.3 will be denoted by  $\{u, v\}$  in the sequel. This FSS will allow us to define the main tools of the present research — the Green function  $G(x, t)$  and the Green integral operator  $G$ :

$$G(x, t) = \begin{cases} u(x)v(t), & x \geq t, \\ u(t)v(x), & x \leq t, \end{cases} \quad x, t \in \mathbb{R} \quad (1.12)$$

$$(Gf)(x) = \int_{-\infty}^{\infty} G(x, t)f(t)dt, \quad f \in L_p, \quad x \in \mathbb{R}. \quad (1.13)$$

**Theorem 1.4.** *Suppose that conditions (1.2) and (1.5) hold, and let  $p \in [1, \infty)$ . Then equation (1.1) is correctly solvable in  $L_p$  if and only if the operator  $G : L_p \rightarrow L_p$  is bounded.*

**Corollary 1.5.** *Suppose that conditions (1.2) and (1.5) hold and equation (1.1) is correctly solvable in  $L_p$ ,  $p \in [1, \infty)$ . Then for any function  $f \in L_p$  the solution  $y \in L_p$  of (1.1) is of the form*

$$y(x) = \int_{-\infty}^{\infty} G(x, t)f(t)dt, \quad x \in \mathbb{R}. \quad (1.14)$$

We now need the following assertion.

**Lemma 1.6.** [3] *Suppose that conditions (1.2) and (1.6) hold. Then for any given  $x \in \mathbb{R}$  each of the equations*

$$\int_{x-d}^x \frac{dt}{r(t)} \cdot \int_{x-d}^x q(t)dt = 1, \quad \int_x^{x+d} \frac{dt}{r(t)} \cdot \int_x^{x+d} q(t)dt = 1 \quad (1.15)$$

in  $d \geq 0$  has a unique finite positive solution. Denote the solutions of (1.15) by  $d_1(x)$ ,  $d_2(x)$ , respectively. For  $x \in \mathbb{R}$  let us introduce the following auxiliary functions:

$$\varphi(x) = \int_{x-d_1(x)}^x \frac{dt}{r(t)}, \quad \psi(x) = \int_x^{x+d_2(x)} \frac{dt}{r(t)} \quad (1.16)$$

$$h(x) = \frac{\varphi(x)\psi(x)}{\varphi(x) + \psi(x)} \quad \left( \equiv \left( \int_{x-d_1(x)}^{x+d_2(x)} q(t) \right)^{-1} \right). \quad (1.17)$$

Furthermore, for every  $x \in \mathbb{R}$ , the equation  $d \geq 0$  in

$$\int_{x-d}^{x+d} \frac{dt}{r(t)h(t)} = 1 \quad (1.18)$$

has a unique finite positive function. Denote it by  $d(x)$ . The function  $d(x)$  is continuous for  $x \in \mathbb{R}$ . In addition,

$$\lim_{x \rightarrow -\infty} (x + d(x)) = -\infty, \quad \lim_{x \rightarrow +\infty} (x - d(x)) = \infty.$$

*Remark 1.7.* Various auxiliary functions similar to the functions in Lemma 1.6 were introduced by M. Otelbaev (see [12]).

Let us now state the main result of the present paper.

**Theorem 1.8.** *Suppose conditions (1.2) and (1.4) hold. Then equation (1.1) is correctly solvable in  $L_p$ ,  $p \in (1, \infty)$  if and only if condition (1.5) holds and  $B < \infty$ . Here*

$$B = \sup_{x \in \mathbb{R}} h(x)d(x). \quad (1.19)$$

Moreover, this criterion reduces to the unique condition  $B < \infty$  if condition (1.4) is replaced with condition (1.6). Finally, in any case one of the following assertions holds:

- $\alpha)$  for every  $p \in (1, \infty)$ , equation (1.1) is correctly solvable in  $L_p$ ;
- $\beta)$  for all  $p \in (1, \infty)$ , equation (1.1) is not correctly solvable in  $L_p$ .

**Corollary 1.9.** [3] *Suppose conditions (1.2) and (1.6) hold. Then for every  $p \in (1, \infty)$ , equation (1.1) is correctly solvable in  $L_p$  if  $A > 0$ . Here*

$$A = \inf_{x \in \mathbb{R}} \frac{1}{2d(x)} \int_{x-d(x)}^{x+d(x)} q(t)dt. \quad (1.20)$$

**Corollary 1.10.** [5] *Let  $r \equiv 1$ , and suppose that the function  $q$  satisfies condition (1.2). Then equation (1.1) is correctly solvable in  $L_p$ ,  $p \in (1, \infty)$  if and only if there exists  $a \in (0, \infty)$  such that  $m(a) > 0$ . Here*

$$m(a) = \inf_{x \in \mathbb{R}} \int_{x-a}^{x+a} q(t)dt. \quad (1.21)$$

*Remark 1.11.* Under some additional assumptions on the functions  $\varphi$  and  $\psi$ , Theorem 1.8 was obtained in [4]. Corollary 1.9 remains true also for  $p = 1$  (see [3]). Corollary 1.10 remains true for  $p = 1$  and  $p = \infty$  (here  $L_\infty(\mathbb{R}) := C(\mathbb{R})$ , see [5]).

*Remark 1.12.* See §3 for the proofs of all the above statements. In §4 we give examples of applications of Theorem 1.8 to concrete equations. §2 contains a list of various facts used in §§3–4.

To conclude this introductory section, note that the results and methods presented here allow us to find exact conditions for:

- 1) correct solvability of equation (1.1) in the spaces  $L_1(\mathbb{R})$  and  $C(\mathbb{R})$ ;
- 2) compactness of the operator  $\mathcal{L}_p^{-1} : L_p \rightarrow L_p$  for  $p \in [1, \infty)$ ;
- 3) separability of equation (1.1) in  $L_p$ ,  $p \in (1, \infty)$  (the problem of Everitt-Giertz, see [9, 10, 2]; and see [13, 3] for the case  $p = 1$ ). The solutions of these problems will appear in our forthcoming papers.

## 2. PRELIMINARIES

**Theorem 2.1.** [8] *Under conditions (1.2) and (1.5) on the FSS  $\{u, v\}$  of equation (1.7) and the Green function  $G(x, t)$  (see (1.12)), the Davies-Harrell representations hold:*

$$\begin{aligned} u(x) &= \sqrt{\rho(x)} \exp \left( -\frac{1}{2} \int_{x_0}^x \frac{dt}{r(t)\rho(t)} \right), \quad x \in \mathbb{R} \\ v(x) &= \sqrt{\rho(x)} \exp \left( \frac{1}{2} \int_{x_0}^x \frac{dt}{r(t)\rho(t)} \right), \quad x \in \mathbb{R} \end{aligned} \quad (2.1)$$

$$G(x, t) = \sqrt{\rho(x)\rho(t)} \exp \left( -\frac{1}{2} \left| \int_t^x \frac{d\xi}{r(\xi)\rho(\xi)} \right| \right). \quad (2.2)$$

Here  $x, t \in \mathbb{R}$ ,  $x_0$  is the unique solution of the equation  $u(x) = v(x)$  in  $\mathbb{R}$  and

$$\rho(x) = u(x)v(x), \quad x \in \mathbb{R}. \quad (2.3)$$

*Remark 2.2.* Representations (2.1) and (2.2) were obtained in [8] for  $r \equiv 1$ . Theorem 2.1 was proved in [3].

**Lemma 2.3.** [3] *Suppose conditions (1.2) and (1.6) hold. Then for  $x \in \mathbb{R}$  we have the following inequalities (see (1.7) and (1.16)):*

$$\begin{aligned} 2^{-1}v(x) &\leq r(x)v'(x)\varphi(x) \leq 2v(x), \\ 2^{-1}u(x) &\leq r(x)|u'(x)|\psi(x) \leq 2u(x). \end{aligned} \quad (2.4)$$

*Remark 2.4.* Two-sided, sharp by order, a priori estimates of type (2.4) first appeared in [18] (for  $r \equiv 1$  and some additional requirements to  $q$ ). Under the conditions (1.2) and  $\inf_{x \in \mathbb{R}} q(x) > 0$ , estimates similar to (2.4), with other, more complicated, auxiliary functions, were given in [13].

**Lemma 2.5.** [3] *Suppose conditions (1.1) and (1.6) hold. The for  $x \in \mathbb{R}$ , we have the inequalities (see (1.17) and (2.3)):*

$$2^{-1}h(x) \leq \rho(x) \leq 2h(x). \quad (2.5)$$

**Lemma 2.6.** [3] *Let  $r \equiv 1$  and suppose that  $q$  satisfies conditions (1.1) and (1.5). Then for every given  $x \in \mathbb{R}$  the equation*

$$d \int_{x-d}^{x+d} q(t) dt = 2 \quad (2.6)$$

*in  $d \geq 0$  has a unique finite positive solution. Denote it by  $\tilde{d}(x)$ . We have the inequalities (see (2.3)):*

$$4^{-1}\tilde{d}(x) \leq \rho(x) \leq 3 \cdot 2^{-1}\tilde{d}(x), \quad x \in \mathbb{R}. \quad (2.7)$$

*Remark 2.7.* Two-sided, sharp by order, a priori estimates of the function  $\rho$  first appeared in [16] (under some additional requirements to  $r$  and  $q$ ). Therefore, we call all inequalities of such a form Otelbaev inequalities. Note that in [16] there were used other, more complicated auxiliary functions than  $h(x)$  and  $\tilde{d}(x)$ . The function  $\tilde{d}(x)$  was introduced by M. Otelbaev (see [12]).

**Lemma 2.8.** [3] *Suppose conditions (1.2) and (1.6) hold. Then for all  $x \in \mathbb{R}$  and  $t \in [x - d(x), x + d(x)]$  (see (1.18)), we have the inequalities (see (2.3)):*

$$e^{-2}\rho(x) \leq \rho(t) \leq e^2\rho(x). \quad (2.8)$$

**Definition 2.9.** [3] *We say that a system of segments  $\{\Delta_n\}_{n \in \mathbb{N}'}$ ,  $\mathbb{N}' = \{\pm 1, \pm 2, \dots\}$  forms an  $\mathbb{R}(x)$ -covering of  $\mathbb{R}$  if the following assertions hold (see (1.18)):*

- 1)  $\Delta_n = [\Delta_n^-, \Delta_n^+] \stackrel{\text{def}}{=} [x_n - d(x_n), x_n + d(x_n)], \quad n \in \mathbb{N}'$ ;
- 2)  $\Delta_{n+1}^- = \Delta_n^+, \quad \text{if } n \geq 1; \quad \Delta_{n-1}^+ = \Delta_n^- \quad \text{if } n \leq -1$ ;
- 3)  $\Delta_1^- = \Delta_{-1}^+ = x, \quad \bigcup_{n \neq 0} \Delta_n = \mathbb{R}$ .

**Lemma 2.10.** [3] *Under conditions (1.2) and (1.6), for every  $x \in \mathbb{R}$ , there exists an  $\mathbb{R}(x)$ -covering of  $\mathbb{R}$ .*

*Remark 2.11.* Assertions similar to Lemma 2.10 were introduced and systematically studied by M. Otelbaev (see [12]).

**Lemma 2.12.** [3] *Suppose conditions (1.2) and (1.5) hold. Then equation (1.7) has no solutions  $z \in L_p$ ,  $p \in [1, \infty)$  apart from  $z \equiv 0$ .*

**Theorem 2.13.** *Let  $\mu, \theta$  be continuous positive functions in  $\mathbb{R}$ , and let  $\mathcal{K}$  be an integral operator:*

$$(\mathcal{K}f)(t) = \mu(t) \int_t^\infty \theta(\xi) f(\xi) d\xi, \quad t \in \mathbb{R}. \quad (2.9)$$

Then for  $p \in (1, \infty)$  the operator  $\mathcal{K} : L_p \rightarrow L_p$  is bounded if and only if  $H_p < \infty$ . Here  $H_p = \sup_{x \in \mathbb{R}} H_p(x)$ ,

$$H_p(x) = \left[ \int_{-\infty}^x \mu(t)^p dt \right]^{1/p} \left[ \int_x^{\infty} \theta(t)^{p'} dt \right]^{1/p'}, \quad p' = \frac{p}{p-1}, \quad x \in \mathbb{R}. \quad (2.10)$$

Moreover, the following inequalities hold:

$$H_p \leq \|\mathcal{K}\|_{p \rightarrow p} \leq (p)^{1/p} (p')^{1/p'} H_p. \quad (2.11)$$

**Theorem 2.14.** Let  $\mu, \theta$  be continuous positive functions in  $\mathbb{R}$ , and let  $\tilde{\mathcal{K}}$  be an integral operator:

$$(\tilde{\mathcal{K}}f)(t) = \mu(t) \int_{-\infty}^t \theta(\xi) f(\xi) d\xi, \quad t \in \mathbb{R}. \quad (2.12)$$

Then for  $p \in (1, \infty)$  the operator  $\tilde{\mathcal{K}} : L_p \rightarrow L_p$  is bounded if and only if  $\tilde{H}_p < \infty$ . Here  $\tilde{H}_p = \sup_{x \in \mathbb{R}} \tilde{H}_p(x)$ ,

$$\tilde{H}_p(x) = \left[ \int_{-\infty}^x \theta(t)^{p'} dt \right]^{1/p'} \left[ \int_x^{\infty} \mu(t)^p dt \right]^{1/p}, \quad p' = \frac{p}{p-1}, \quad x \in \mathbb{R}. \quad (2.13)$$

Moreover, the following inequalities hold:

$$\tilde{H}_p \leq \|\tilde{\mathcal{K}}\|_{p \rightarrow p} \leq (p)^{1/p} (p')^{1/p'} \tilde{H}_p. \quad (2.14)$$

*Remark 2.15.* Theorems 2.13 and 2.14 follow from a Hardy type inequality (see [15]). In particular, see [6] for such a proof. See [17] for the original direct proof of these theorems (under weaker requirements of  $\mu$  and  $\theta$ ).

**Theorem 2.16.** [11, Ch.V, §2, no.5] Let  $-\infty \leq a < b \leq \infty$ , let  $\mathcal{K}(s, t)$  be a continuous function for  $s, t \in [a, b]$ , and let  $\mathcal{K}$  be an integral operator

$$(\mathcal{K}f)(t) = \int_a^b \mathcal{K}(s, t) f(s) ds, \quad t \in [a, b]. \quad (2.15)$$

Then

$$\|\mathcal{K}\|_{L_1(a, b) \rightarrow L_1(a, b)} = \sup_{s \in [a, b]} \int_a^b |\mathcal{K}(s, t)| dt. \quad (2.16)$$

### 3. PROOFS

*Proof of Theorem 1.1.* Assume the contrary. Then for some  $p \in [1, \infty)$ , equation (1.1) is correctly solvable in  $L_p$ , and there exists a point  $x_0 \in \mathbb{R}$  such that

$$\int_{x_0}^{\infty} q(t) dt = 0 \quad \Rightarrow \quad q(t) = 0 \quad \text{almost everywhere on } (x_0, \infty). \quad (3.1)$$

Let  $x_1 \gg \max\{1, |x_0|\}$ , and let

$$f(x) = \begin{cases} -1, & \text{if } x \in [x_0, x_1] \\ 0, & \text{if } x \notin [x_0, x_1] \end{cases} \Rightarrow \|f\|_p = (x_1 - x_0)^{1/p}. \quad (3.2)$$

By (3.1) and (3.2), for  $x \geq x_1$ , equation (1.1) is of the form

$$-(r(x)y'(x))' = 0, \quad x \geq x_1 \Rightarrow \quad (3.3)$$

$$y(x) = c_1 + c_2 \int_{x_1}^x \frac{dt}{r(t)}, \quad x \geq x_1. \quad (3.4)$$

Here  $c_1, c_2$  are some constants. From I)–II) (see §1) it follows that

$$\|y\|_{L_p(x_1, \infty)} \leq \|y\|_p \leq c(p)\|f\|_p < \infty. \quad (3.5)$$

Then, according to (3.5) and (1.4), we conclude that  $c_1 = c_2 = 0$ . Indeed, if  $c_2 \neq 0$ , then for all  $x \gg x_1$  we get

$$|y(x)| \geq |c_2| \int_{x_1}^x \frac{dt}{r(t)} \left[ 1 - \left| \frac{c_1}{c_2} \right| \left( \int_{x_1}^x \frac{dt}{r(t)} \right)^{-1} \right] \geq \frac{|c_2|}{2} > 0$$

and therefore  $\|y\|_p = \infty$ . Contradiction. Hence  $c_2 = 0$ . But then also  $c_1 = 0$  because otherwise  $y = c_1 \notin L_p$ . Thus  $y(x) \equiv 0$  for  $x \geq x_1$ . Hence in the case (3.2), equation (1.1) has solution  $y \in L_p$  which, for  $x \in [x_0, x_1]$ , satisfies the relations

$$-(r(x)y'(x))' = -1, \quad y(x_1) = y'(x_1) = 0 \Rightarrow y(x) = \int_x^{x_1} \frac{x_1 - t}{r(t)} dt, \quad x \in [x_0, x_1]. \quad (3.6)$$

Let us first consider the case  $p = 1$ . From (3.6) it follows that

$$\begin{aligned} \|y\|_{L_1(x_0, x_1)} &= \int_{x_0}^{x_1} \left( \int_x^{x_1} \frac{x_1 - t}{r(t)} dt \right) dx = (x - x_0) \int_x^{x_1} \frac{x_1 - t}{r(t)} dt \Big|_{x_0}^{x_1} + \int_{x_0}^{x_1} \frac{(x - x_0)(x_1 - x)}{r(x)} dx \\ &= \int_{x_0}^{x_1} \frac{(x - x_0)(x_1 - x)}{r(x)} dx. \end{aligned} \quad (3.7)$$

Since  $x_1 \gg \max\{1, |x_0|\}$ , according to (3.7) we get

$$\begin{aligned} \|y\|_{L_1(x_0, x_1)} &\geq \int_{x_0+1}^{x_1-1} \frac{(x - x_0)(x_1 - x)}{r(x)} dx \geq (x_1 - x_0 - 1) \int_{x_0+1}^{x_1-1} \frac{dx}{r(x)} \\ &\geq \frac{x_1 - x_0}{2} \int_{x_0+1}^{x_1-1} \frac{dx}{r(x)}. \end{aligned} \quad (3.8)$$

But then from (3.8) and (1.3), it follows that

$$\begin{aligned} \frac{x_1 - x_0}{2} \int_{x_0+1}^{x_1-1} \frac{dx}{r(x)} &\leq \|y\|_{L_1(x_0, x_1)} \leq \|y\|_1 \leq c(1)\|f\|_1 = c(1)(x_1 - x_0) \Rightarrow \\ &\int_{x_0+1}^{x_1-1} \frac{dx}{r(x)} \leq 2c(1) < \infty. \end{aligned} \quad (3.9)$$

Since  $x_1$  can be chosen arbitrarily large, (3.9) contradicts (1.4), and in the case  $p = 1$  the theorem is proven.



Consider the case  $p \in (1, \infty)$ . From (3.6), for  $x \in [x_0, x_0 + 1]$ , it follows that

$$\begin{aligned} y(x) &= \int_x^{x_0+1} \frac{x_1 - t}{r(t)} dt + \int_{x_0+1}^{x_1} \frac{x_1 - t}{r(t)} dt > \int_{x_0+1}^{x_1} \frac{x_1 - t}{r(t)} dt > \int_{x_0+1}^{x_0+2} \frac{x_1 - t}{r(t)} dt \\ &\geq \int_{x_0+1}^{x_0+2} \frac{x_1 - x_0 - 2}{r(t)} dt \geq \frac{x_1 - x_0}{2} \int_{x_0+1}^{x_0+2} \frac{dt}{r(t)} \Rightarrow \\ \|y\|_{L_p(x_0, x_0+1)} &\geq \frac{x_1 - x_0}{2} \int_{x_0+1}^{x_0+2} \frac{dt}{r(t)}. \end{aligned} \quad (3.10)$$

Then, according to (3.10) and (1.3), we get

$$\begin{aligned} \frac{x_1 - x_0}{2} \int_{x_0+1}^{x_0+2} \frac{dt}{r(t)} &\leq \|y\|_{L_p(x_0, x_0+1)} \leq \|y\|_p \leq c(p) \|f\|_p = c(p) (x_1 - x_0)^{1/p} \\ \Rightarrow (x_1 - x_0)^{1/p'} \int_{x_0+1}^{x_0+2} \frac{dt}{r(t)} &\leq 2c(p) < \infty, \quad p' = \frac{p}{p-1}. \end{aligned} \quad (3.11)$$

Since  $x_1$  can be taken arbitrarily large, (3.11) contradicts (1.2). Thus inequalities (1.5) are proven. It remains to notice that (1.6) follows from (1.5) and (1.4).  $\square$

*Proof of Theorem 1.4. Necessity.*

We need the following (maybe commonly known) assertion.

**Lemma 3.1.** *Suppose conditions (1.2) and (1.5) hold. Consider the integral equations*

$$(G_1 f)(x) = u(x) \int_{-\infty}^x v(t) f(t) dt, \quad x \in \mathbb{R}; \quad (3.12)$$

$$(G_2 f)(x) = v(x) \int_x^{\infty} u(t) f(t) dt, \quad x \in \mathbb{R}; \quad (3.13)$$

For  $p \in [1, \infty)$  the following inequalities hold:

$$\frac{\|G_1\|_{p \rightarrow p} + \|G_2\|_{p \rightarrow p}}{2} \leq \|G\|_{p \rightarrow p} \leq \|G_1\|_{p \rightarrow p} + \|G_2\|_{p \rightarrow p}. \quad (3.14)$$

*Proof.* The upper estimate for  $\|G\|_{p \rightarrow p}$  follows from the triangle inequality for norms. Furthermore, the following relations are obvious:

$$\begin{aligned} \|G_1 f\|_p &= \left[ \int_{-\infty}^{\infty} u(x)^p \left| \int_{-\infty}^x v(t) f(t) dt \right|^p dx \right]^{1/p} \\ &\leq \left[ \int_{-\infty}^{\infty} u(x)^p \left( \int_{-\infty}^x v(t) |f(t)| dt \right)^p dx \right]^{1/p} \\ &\leq \left\{ \int_{-\infty}^{\infty} \left[ u(x) \int_{-\infty}^x v(t) |f(t)| dt + v(x) \int_x^{\infty} u(t) |f(t)| dt \right]^p dx \right\}^{1/p} \\ &= \|G|f|\|_p \leq \|G\|_{p \rightarrow p} \cdot \|f\|_p \Rightarrow \|G_1\|_{p \rightarrow p} \leq \|G\|_{p \rightarrow p}. \end{aligned}$$

In a similar way, one can check that  $\|G_2\|_{p \rightarrow p} \leq \|G\|_{p \rightarrow p}$ . These inequalities imply the lower estimate of (3.14).  $\square$

Let us now go to the proof of the theorem. Suppose that equation (1.1) is correctly solvable in  $L_p$  for some  $p \in [1, \infty)$ . Take an arbitrary pair of numbers  $x_1, x_2$  ( $x_1 \leq x_2$ ), and for any function  $f \in L_p$  set

$$\tilde{f}(x) = \begin{cases} f(x), & \text{if } x \in [x_1, x_2] \\ 0, & \text{if } x \notin [x_1, x_2] \end{cases}$$

Let  $\tilde{y}$  be a solution of (1.1) with right-hand side  $\tilde{f}$  such that  $\tilde{y} \in L_p$ . It is easy to see that  $\tilde{y}$  is of the form

$$\begin{aligned} \tilde{y}(x) = & c_1 u(x) + c_2 v(x) + u(x) \int_{-\infty}^x v(t) \tilde{f}(t) dt \\ & + v(x) \int_x^{\infty} u(t) \tilde{f}(t) dt, \quad x \in \mathbb{R}. \end{aligned} \quad (3.15)$$

Here  $c_1, c_2$  are some constants. Indeed, if the integrals in (3.15) exist, then representation (3.15) holds true for  $\tilde{y}$  because this formula represents the general solution of (1.1). The existence of these integrals follows from the definition of  $\tilde{f}$ . For example, for  $p \in (1, \infty)$  and  $x \in \mathbb{R}$ , using Hölder's inequality and the definition of  $\tilde{f}$ , we obtain

$$\begin{aligned} \int_{-\infty}^x v(t) |\tilde{f}(t)| dt &\leq \int_{x_1}^{x_2} v(t) |\tilde{f}(t)| dt \leq \left( \int_{x_1}^{x_2} v(t)^{p'} dt \right)^{1/p'} \|f\|_p, \quad p' = \frac{p}{p-1}; \\ \int_{-\infty}^x u(t) |\tilde{f}(t)| dt &\leq \int_{x_1}^{x_2} u(t) |\tilde{f}(t)| dt \leq \left( \int_{x_1}^{x_2} u(t)^{p'} dt \right)^{1/p'} \|f\|_p, \quad p' = \frac{p}{p-1}. \end{aligned} \quad (3.16)$$

Thus, the integrals exist and formula (3.15) holds true.

Let us now prove that  $c_1 = c_2 = 0$ . Assume the contrary. Let, say,  $c_2 \neq 0$ . Then for  $x \geq x_2$ , we have

$$\begin{aligned} |\tilde{y}(x)| &\geq |c_2| v(x) - |c_1| u(x) - u(x) \int_{x_1}^{x_2} v(t) |\tilde{f}(t)| dt \\ &= |c_2| v(x) \left[ 1 - \left| \frac{c_1}{c_2} \right| \frac{u(x)}{v(x)} - \frac{1}{|c_2|} \frac{u(x)}{v(x)} \int_{x_1}^{x_2} v(t) |\tilde{f}(t)| dt \right]. \end{aligned}$$

The latter inequality, together with (3.16) and (1.10), implies

$$|\tilde{y}(x)| \geq \frac{|c_2|}{2} v(x), \quad x \gg \max\{1, |x_2|\} \Rightarrow \|\tilde{y}\|_p = \infty,$$

a contradiction. Hence  $c_2 = 0$  and, similarly,  $c_1 = 0$ , and therefore

$$\tilde{y}(x) = \int_{-\infty}^{\infty} G(x, t) \tilde{f}(t) dt = \int_{x_1}^{x_2} G(x, t) \tilde{f}(t) dt, \quad x \in \mathbb{R}. \quad (3.17)$$

Below we consider the cases  $p = 1$  and  $p \in (1, \infty)$  separately. For  $p = 1$ , using (3.17) and (1.3), we get

$$\|\tilde{y}\|_{L_1(x_1, x_2)} \leq \|\tilde{y}\|_1 \leq c(1) \|\tilde{f}\|_1 = c(1) \|\tilde{f}\|_{L_1(x_1, x_2)}.$$

Hence the operator  $G : L_1(x_1, x_2) \rightarrow L_1(x_1, x_2)$  (see (3.17)) is bounded, and its norm does not exceed  $c(1)$  (see Theorem 2.16), i.e.:

$$\|G\|_{L_1(x_1, x_2) \rightarrow L_1(x_1, x_2)} = \sup_{t \in [x_1, x_2]} \int_{x_1}^{x_2} G(x, t) dx \leq c(1). \quad (3.18)$$

Since  $x_1, x_2$  are arbitrary numbers, (3.18) implies the inequality

$$\sup_{t \in \mathbb{R}} \int_{-\infty}^{\infty} G(x, t) dx \leq c(1).$$

Then, using Theorem 2.16 once again, we get

$$\|G\|_{1 \rightarrow 1} = \sup_{t \in \mathbb{R}} \int_{-\infty}^{\infty} G(x, t) dx \leq c(1) < \infty.$$

Q.E.D.

In the case  $p \in (1, \infty)$ , set

$$\begin{aligned} \tilde{f}(x) &= \begin{cases} u(x)^{p'-1} & \text{if } x \in [x_1, x_2], \\ 0 & \text{if } x \notin [x_1, x_2] \end{cases} \quad p' = \frac{p}{p-1} \Rightarrow \\ \|\tilde{f}\|_p &= \left( \int_{x_1}^{x_2} u(x)^{p(p'-1)} dx \right)^{1/p} = \left( \int_{x_1}^{x_2} u(x)^{p'} dx \right)^{1/p} < \infty. \end{aligned} \quad (3.19)$$

Furthermore, from (3.17) it follows that

$$\begin{aligned} \tilde{y}(x) &= \int_{-\infty}^{\infty} G(x, t) \tilde{f}(t) dt = u(x) \int_{-\infty}^x v(t) \tilde{f}(t) dt + v(x) \int_x^{\infty} u(t) \tilde{f}(t) dt \\ &\geq v(x) \int_x^{\infty} u(t) \tilde{f}(t) dt \Rightarrow \\ \|\tilde{y}\|_p^p &\geq \int_{-\infty}^{\infty} v(x)^p \left[ \int_x^{\infty} u(t) \tilde{f}(t) dt \right]^p dx \geq \int_{-\infty}^{x_1} v(x)^p \left[ \int_x^{\infty} u(t) \tilde{f}(t) dt \right]^p dx \\ &\geq \left( \int_{-\infty}^{x_1} v(x)^p dx \right) \left( \int_{x_1}^{\infty} u(t) \tilde{f}(t) dt \right)^p = \left( \int_{-\infty}^{x_1} v(x)^p dx \right) \left( \int_{x_1}^{x_2} u(t)^{p'} dt \right)^p \Rightarrow \\ \|\tilde{y}\|_p &\geq \left[ \int_{-\infty}^{x_1} v(t)^p dt \right]^{1/p} \left[ \int_{x_1}^{x_2} u(t)^{p'} dt \right]. \end{aligned} \quad (3.20)$$

From (1.3), (3.19) and (3.20), we now get

$$\begin{aligned} \left[ \int_{-\infty}^{x_1} v(t)^p dt \right]^{1/p} \left[ \int_{x_1}^{x_2} u(t)^{p'} dt \right] &\leq \|\tilde{y}\|_p \leq c(p) \|\tilde{f}\|_p \\ &= c(p) \left[ \int_{x_1}^{x_2} u(t)^{p'} dt \right]^{1/p} \Rightarrow \\ \left[ \int_{-\infty}^{x_1} v(t)^p dt \right]^{1/p} \left[ \int_{x_1}^{x_2} u(t)^{p'} dt \right]^{1/p'} &\leq c(p) < \infty. \end{aligned} \quad (3.21)$$

Since in (3.21) the numbers  $x_1, x_2$  are arbitrary, (3.21) implies

$$\sup_{x \in \mathbb{R}} \left[ \int_{-\infty}^x v(t)^p dt \right]^{1/p} \left[ \int_x^{\infty} u(t)^{p'} dt \right]^{1/p'} \leq c(p) < \infty.$$

By Theorem 2.13, this implies that the operator  $G_2 : L_p \rightarrow L_p$  (see (3.13)) is bounded. Similarly, using Theorem 2.14, we conclude that the operator  $G_1 : L_p \rightarrow L_p$  (see (3.12)) is bounded, too. Hence the operator  $G : L_p \rightarrow L_p$  is bounded in view of Lemma 3.1.

*Proof of Theorem 1.4.* Sufficiency.

We shall only prove the theorem in the case  $p \in (1, \infty)$  since for  $p = 1$  the argument is simpler and goes along similar lines. So let  $\|G\|_{p \rightarrow p} < \infty$ . Then  $\|G_1\|_{p \rightarrow p}, \|G_2\|_{p \rightarrow p} < \infty$  by Lemma 3.1, and therefore from Theorems 2.13 and 2.14 we obtain the inequalities:

$$\int_{-\infty}^x v(t)^{p'} dt < \infty, \quad \int_x^{\infty} u(t)^{p'} dt < \infty \quad \forall x \in \mathbb{R}.$$

Together with Hölder's inequality this implies that for any  $x \in \mathbb{R}$  and any function  $f \in L_p$ , there exist the integrals

$$\int_{-\infty}^x v(t)f(t)dt, \quad \int_x^{\infty} u(t)f(t)dt.$$

Thus for  $x \in \mathbb{R}$  the following function is well-defined:

$$y(x) = u(x) \int_{-\infty}^x v(t)f(t)dt + v(x) \int_x^{\infty} u(t)f(t)dt \equiv (Gf)(x).$$

A straightforward computation shows that  $y = Gf$  is a solution of (1.1) for which inequality (1.3) holds because  $\|G\|_{p \rightarrow p} < \infty$ . To complete the proof of the theorem, it remains to apply Lemma 2.12.  $\square$

*Proof of Corollary 1.5.* If equation (1.1) is correctly solvable in  $L_p$ ,  $p \in [1, \infty)$ , then the operator  $G : L_p \rightarrow L_p$  is bounded in view of Theorem 1.4. At this point, to prove the corollary, it is enough to repeat the argument from the “sufficiency part” of Theorem 1.4.  $\square$

*Proof of Theorem 1.8.* Necessity.

Below we need the following simple fact.

**Lemma 3.2.** *Suppose we are under the conditions of Theorem 1.8. Consider the functions*

$$\Phi_1(x) = \left[ \int_{-\infty}^x v(t)^{p'} dt \right]^{1/p'} \cdot \left[ \int_x^{\infty} u(t)^p dt \right]^{1/p}, \quad x \in \mathbb{R}; \quad (3.22)$$

$$\Phi_2(x) = \left[ \int_{-\infty}^x v(t)^p dt \right]^{1/p} \cdot \left[ \int_x^{\infty} u(t)^{p'} dt \right]^{1/p'}, \quad x \in \mathbb{R}, \quad (3.23)$$

where  $p \in (1, \infty)$ ,  $p' = \frac{p}{p-1}$ . Then for  $x \in \mathbb{R}$ , we have

$$\begin{aligned} \Phi_1(x) &= \left[ \int_{-\infty}^x \rho(t)^{p'/2} \exp \left( -\frac{p'}{2} \int_t^x \frac{d\xi}{r(\xi)\rho(\xi)} \right) dt \right]^{1/p'} \\ &\quad \cdot \left[ \int_x^\infty \rho(t)^{p/2} \exp \left( -\frac{p}{2} \int_x^t \frac{d\xi}{r(\xi)\rho(\xi)} \right) dt \right]^{1/p} \end{aligned} \quad (3.24)$$

$$\begin{aligned} \Phi_2(x) &= \left[ \int_{-\infty}^x \rho(t)^{p/2} \exp \left( -\frac{p}{2} \int_t^x \frac{d\xi}{r(\xi)\rho(\xi)} \right) dt \right]^{1/p} \\ &\quad \cdot \left[ \int_x^\infty \rho(t)^{p'/2} \exp \left( -\frac{p'}{2} \int_x^t \frac{d\xi}{r(\xi)\rho(\xi)} \right) dt \right]^{1/p'}. \end{aligned} \quad (3.25)$$

*Proof.* Equalities (3.24) and (3.25) are proved in the same way. For example, let us prove (3.25). Let us substitute (2.1) into (3.23):

$$\begin{aligned} \Phi_2(x) &= \left[ \int_{-\infty}^x \rho(t)^{p/2} \exp \left( \frac{p}{2} \int_{x_0}^t \frac{d\xi}{r(\xi)\rho(\xi)} \right) dt \right]^{1/p} \left[ \int_x^\infty \rho(t)^{p'/2} \exp \left( -\frac{p'}{2} \int_{x_0}^t \frac{d\xi}{r(\xi)\rho(\xi)} \right) dt \right]^{1/p'} \\ &= \left[ \int_{-\infty}^x \rho(t)^{p/2} \exp \left( -\frac{p}{2} \int_t^x \frac{d\xi}{r(\xi)\rho(\xi)} \right) \cdot \exp \left( \frac{p}{2} \int_{x_0}^x \frac{d\xi}{r(\xi)\rho(\xi)} \right) dt \right]^{1/p} \\ &\quad \cdot \left[ \int_x^\infty \rho(t)^{p'/2} \exp \left( -\frac{p'}{2} \int_x^t \frac{d\xi}{r(\xi)\rho(\xi)} \right) \exp \left( -\frac{p'}{2} \int_{x_0}^x \frac{d\xi}{r(\xi)\rho(\xi)} \right) dt \right]^{1/p'} \\ &= \left[ \int_{-\infty}^x \rho(t)^{p/2} \exp \left( -\frac{p}{2} \int_t^x \frac{d\xi}{r(\xi)\rho(\xi)} \right) dt \right]^{1/p} \\ &\quad \cdot \left[ \int_x^\infty \rho(t)^{p'/2} \exp \left( -\frac{p'}{2} \int_x^t \frac{d\xi}{r(\xi)\rho(\xi)} \right) dt \right]^{1/p'}. \end{aligned}$$

□

Let us now go to the proof of the theorem. Suppose conditions (1.2) and (1.4) hold, and for some  $p \in (1, \infty)$  equation (1.1) is correctly solvable in  $L_p$ . Then by Theorem 1.1, relations (1.5) and (1.6) hold. Hence all auxiliary functions from Lemma 1.6 are well-defined. Furthermore, by Theorem 1.4 the operator  $G : L_p \rightarrow L_p$  is bounded, and therefore the operator  $G_2 : L_p \rightarrow L_p$  (see Lemma 3.1) is bounded, too. In the following relations, we consecutively use (3.14), (2.11), (3.25), Lemma 1.6, (2.8), (2.5) and (1.18):

$$\begin{aligned}
\infty &> 2\|G\|_{p \rightarrow p} \geq \|G_2\|_{p \rightarrow p} \geq \sup_{x \in \mathbb{R}} \left[ \int_{-\infty}^x v(t)^p dt \right]^{1/p} \left[ \int_x^{\infty} u(t)^{p'} dt \right]^{1/p'} \\
&= \sup_{x \in \mathbb{R}} \left[ \int_{-\infty}^x \rho(t)^{p/2} \exp \left( -\frac{p}{2} \int_t^x \frac{d\xi}{r(\xi)\rho(\xi)} \right) dt \right]^{1/p} \\
&\quad \cdot \left[ \int_x^{\infty} \rho(t)^{p'/2} \exp \left( -\frac{p'}{2} \int_x^t \frac{d\xi}{r(\xi)\rho(\xi)} \right) dt \right]^{1/p'} \\
&\geq \sup_{x \in \mathbb{R}} \left[ \int_{x-d(x)}^x \rho(t)^{p/2} \exp \left( -\frac{p}{2} \int_t^x \frac{d\xi}{r(\xi)\rho(\xi)} \right) dt \right]^{1/p} \\
&\quad \cdot \left[ \int_x^{x+d(x)} \rho(t)^{p'/2} \exp \left( -\frac{p'}{2} \int_x^t \frac{d\xi}{r(\xi)\rho(\xi)} \right) dt \right]^{1/p'} \\
&\geq \sup_{x \in \mathbb{R}} e^{-4} \rho(x) \exp \left( -\frac{1}{2} \int_{x-d(x)}^{x+d(x)} \frac{d\xi}{r(\xi)\rho(\xi)} \right) d(x)^{\frac{1}{p} + \frac{1}{p'}} \\
&\geq 2^{-1} e^{-5} \sup_{x \in \mathbb{R}} h(x) d(x) = 2^{-1} e^{-5} B.
\end{aligned}$$

*Proof of Theorem 1.8.* Sufficiency.

Suppose conditions (1.2), (1.4), (1.5) hold, and let  $p \in (1, \infty)$ . Then equality (1.6) evidently holds, and therefore all the auxiliary functions from Lemma 1.6 are well-defined. Let us show that the condition  $B < \infty$  guarantees that the operator  $G : L_p \rightarrow L_p$  is bounded and thus complete the proof (see Theorem 1.4). By Lemma 3.1, we have  $\|G\|_{p \rightarrow p} < \infty$  if  $\|G_1\|_{p \rightarrow p} < \infty$  and  $\|G_2\|_{p \rightarrow p} < \infty$ . The latter inequalities follow from the estimates

$$\|G_1\|_{p \rightarrow p} \leq c(p)B, \quad \|G_2\|_{p \rightarrow p} \leq cB. \quad (3.26)$$

Both inequalities in (3.26) are checked in the same way. Consider, say, the second one. Let first  $p \in (1, 2]$ ,  $f \in L_p$ , and denote by  $\alpha$  some number from the interval  $[0, 1)$  which will be chosen later. Then by Lemma 1.3, for any  $x \in \mathbb{R}$ , we have

$$\begin{aligned}
|(G_2 f)(x)| &= v(x) \left| \int_x^{\infty} u(t) f(t) dt \right| \leq v(x) \int_x^{\infty} u(t) |f(t)| dt \\
&\leq (v(x) u(x))^\alpha v(x)^{1-\alpha} \int_x^{\infty} u(t)^{1-\alpha} |f(t)| dt \\
&= \rho(x)^\alpha v(x)^{1-\alpha} \int_x^{\infty} u(t)^{1-\alpha} |f(t)| dt, \quad x \in \mathbb{R}.
\end{aligned} \quad (3.27)$$

Consider the integral operator

$$(\mathcal{K}^{(\alpha)} f)(x) = \rho(x)^\alpha v(x)^{1-\alpha} \int_x^{\infty} u(t)^{1-\alpha} f(t) dt, \quad x \in \mathbb{R}. \quad (3.28)$$

Let us show that there exists  $\alpha_0 \in [0, 1)$  such that under the condition  $B < \infty$ , the operator  $\mathcal{K}^{(\alpha)} : L_p \rightarrow L_p$  is bounded.

*Remark 3.3.* Throughout the sequel, including §4, we denote by  $c, c(p)$  some absolute positive constants which are not essential for exposition and may differ within a single chain of calculations.

Below, when estimating  $\|\mathcal{K}^{(\alpha)}\|_{p \rightarrow p}$ , we consecutively use Theorems 2.13 and 2.1:

$$\begin{aligned}
\|\mathcal{K}^{(\alpha)}\|_{p \rightarrow p} &\leq c(p) \sup_{x \in \mathbb{R}} \left[ \int_{-\infty}^x (\rho(t)^\alpha v(t)^{1-\alpha})^p dt \right]^{1/p} \left[ \int_x^\infty (u(t)^{1-\alpha})^{p'} dt \right]^{1/p'} \\
&= c(p) \sup_{x \in \mathbb{R}} \left[ \int_{-\infty}^x \rho(t)^{\frac{1+\alpha}{2}p} \exp \left( -\frac{1-\alpha}{2}p \int_t^x \frac{d\xi}{r(\xi)\rho(\xi)} \right) \exp \left( \frac{1-\alpha}{2}p \int_{x_0}^x \frac{d\xi}{r(\xi)\rho(\xi)} \right) dt \right]^{1/p} \\
&\quad \cdot \left[ \int_x^\infty \rho(t)^{\frac{1-\alpha}{2}p'} \exp \left( -\frac{1-\alpha}{2}p' \int_x^t \frac{d\xi}{r(\xi)\rho(\xi)} \right) \exp \left( -\frac{1-\alpha}{2}p' \int_{x_0}^x \frac{d\xi}{r(\xi)\rho(\xi)} \right) dt \right]^{1/p'} \\
&= c(p) \sup_{x \in \mathbb{R}} \left[ \int_{-\infty}^x \rho(t)^{\frac{1+\alpha}{2}p} \exp \left( -\frac{1-\alpha}{2}p \int_t^x \frac{d\xi}{r(\xi)\rho(\xi)} \right) dt \right]^{1/p} \\
&\quad \cdot \left[ \int_x^\infty \rho(t)^{\frac{1-\alpha}{2}p'} \exp \left( -\frac{1-\alpha}{2}p' \int_x^t \frac{d\xi}{r(\xi)\rho(\xi)} \right) dt \right]^{1/p'}. \tag{3.29}
\end{aligned}$$

Let  $\alpha_0$  be a solution of the equation

$$\frac{1+\alpha}{2}p = \frac{1-\alpha}{2}p' \quad \Rightarrow \quad \alpha_0 = \frac{p' - p}{p' + p} \in [0, 1).$$

Then for  $\alpha = \alpha_0$  the estimate (3.29) can be simplified to the following form:

$$\begin{aligned}
\|\mathcal{K}^{(\alpha_0)}\|_{p \rightarrow p} &\leq c(p) \sup_{x \in \mathbb{R}} \left[ \int_{-\infty}^x \rho(t) \exp \left( -(p-1) \int_t^x \frac{d\xi}{r(\xi)\rho(\xi)} \right) dt \right]^{1/p} \\
&\quad \cdot \left[ \int_x^\infty \rho(t) \exp \left( -\int_x^t \frac{d\xi}{r(\xi)\rho(\xi)} \right) dt \right]^{1/p'}. \tag{3.30}
\end{aligned}$$

Below we continue estimate (3.30) and consecutively apply Lemmas 2.10, 2.8 and 2.5:

$$\begin{aligned}
\|\mathcal{K}^{(\alpha_0)}\|_{p \rightarrow p} &\leq c(p) \sup_{x \in \mathbb{R}} \left[ \sum_{n=-\infty}^{-1} \int_{\Delta_n} \rho(t) \exp \left( -(p-1) \int_t^{\Delta_{-1}^+} \frac{d\xi}{r(\xi)\rho(\xi)} \right) dt \right]^{1/p} \\
&\quad \cdot \left[ \sum_{n=1}^{\infty} \int_{\Delta_n} \rho(t) \exp \left( - \int_{\Delta_1^-}^t \frac{d\xi}{r(\xi)\rho(\xi)} \right) dt \right]^{1/p'} \\
&\leq c(p) \sup_{x \in \mathbb{R}} \left[ \sum_{n=-\infty}^{-1} \rho(x_n) d(x_n) \exp \left( -(p-1) \int_{\Delta_n^+}^{\Delta_{-1}^+} \frac{d\xi}{r(\xi)\rho(\xi)} \right) \right]^{1/p} \\
&\quad \cdot \left[ \sum_{n=1}^{\infty} \rho(x_n) d(x_n) \exp \left( - \int_{\Delta_1^-}^{\Delta_n^-} \frac{d\xi}{r(\xi)\rho(\xi)} \right) \right]^{1/p'} \\
&\leq c(p) \sup_{x \in \mathbb{R}} \left[ \sum_{n=-\infty}^{-1} h(x_n) d(x_n) \exp \left( -\frac{p-1}{2} \int_{\Delta_n^+}^{\Delta_{-1}^+} \frac{d\xi}{r(\xi)h(\xi)} \right) \right]^{1/p} \\
&\quad \cdot \left[ \sum_{n=1}^{\infty} h(x_n) d(x_n) \exp \left( -\frac{1}{2} \int_{\Delta_1^-}^{\Delta_n^-} \frac{d\xi}{r(\xi)h(\xi)} \right) \right]^{1/p'}. \tag{3.31}
\end{aligned}$$

Note that Definition 2.9 and (1.18) imply the equalities

$$\begin{cases} \int_{\Delta_n^+}^{\Delta_{-1}^+} \frac{d\xi}{r(\xi)h(\xi)} = |n| - 1 & \text{if } n \leq -1 \\ \int_{\Delta_1^-}^{\Delta_n^-} \frac{d\xi}{r(\xi)h(\xi)} = n - 1 & \text{if } n \geq 1 \end{cases} \tag{3.32}$$

Equalities (3.32) are checked in the same way, and therefore we only check, say, the second one. For  $n = 1$  it is obvious, and for  $n \geq 2$  we have

$$\int_{\Delta_1^-}^{\Delta_n^-} \frac{d\xi}{r(\xi)h(\xi)} = \sum_{k=1}^{n-1} \int_{\Delta_k} \frac{d\xi}{r(\xi)h(\xi)} = \sum_{k=1}^{n-1} 1 = n - 1 \quad \Rightarrow \quad (3.32).$$



Let us now continue estimate (3.31) using (3.32):

$$\begin{aligned}
\|\mathcal{K}^{(\alpha_0)}\|_{p \rightarrow p} &\leq c(p) \sup_{x \in \mathbb{R}} \left[ \sum_{n=-\infty}^{-1} h(x_n) d(x_n) \exp \left( -\frac{p-1}{2}(|n|-1) \right) \right]^{1/p} \\
&\quad \cdot \left[ \sum_{n=1}^{\infty} h(x_n) d(x_n) \exp \left( -\frac{n-1}{2} \right) \right]^{1/p'} \\
&\leq c(p) B^{\frac{1}{p} + \frac{1}{p'}} \left[ \sum_{n=-\infty}^{-1} \exp \left( -\frac{(p-1)(|n|-1)}{2} \right) \right]^{1/p} \\
&\quad \cdot \left[ \sum_{n=1}^{\infty} \exp \left( -\frac{n-1}{2} \right) \right]^{1/p'} = c(p) B \quad \Rightarrow \\
\|\mathcal{K}^{(\alpha_0)}\|_{p \rightarrow p} &\leq c(p) B \quad \text{for } p \in (1, 2], \quad \alpha_0 = \frac{p' - p}{p' + p}. \tag{3.33}
\end{aligned}$$

Let now  $p \in (2, \infty)$ ,  $f \in L_p$ , and denote by  $\alpha$  some number from the interval  $(0, 1)$  which will be chosen later. Then by Lemma 1.3, for any  $x \in \mathbb{R}$ , we get

$$\begin{aligned}
|(G_2 f)(x)| &\leq v(x) \int_x^{\infty} u(t) |f(t)| dt \leq v(x)^{1-\alpha} \int_x^{\infty} (v(t) u(t))^{\alpha} u(t)^{1-\alpha} |f(t)| dt \\
&= v(x)^{1-\alpha} \int_x^{\infty} \rho(t)^{\alpha} u(t)^{1-\alpha} |f(t)| dt. \tag{3.34}
\end{aligned}$$

Furthermore, in the same way as above, for the norm of the operator

$$(\tilde{\mathcal{K}}^{(\alpha)} f)(x) = v(x)^{1-\alpha} \int_x^{\infty} \rho(t)^{\alpha} u(t)^{1-\alpha} f(t) dt, \quad x \in \mathbb{R}$$

for  $\alpha = \alpha_1 = \frac{p-p'}{p+p'}$ , we establish the estimate

$$\|\tilde{\mathcal{K}}^{(\alpha_0)}\|_{p \rightarrow p} \leq c(p) B, \quad p \in (2, \infty). \tag{3.35}$$

From (3.27) and (3.33), (3.34) and (3.35), it now follows that

$$\|G_2 f\|_p \leq \|\mathcal{K}^{(\alpha_0)} f\|_p \leq \|\mathcal{K}^{(\alpha_0)}\|_{p \rightarrow p} \|f\|_p \leq c(p) B \|f\|_p, \quad p \in (1, 2];$$

$$\|G_2 f\|_p \leq \|\tilde{\mathcal{K}}^{(\alpha_1)} f\|_p \leq \|\tilde{\mathcal{K}}^{(\alpha_1)}\|_{p \rightarrow p} \|f\|_p \leq c(p) B \|f\|_p, \quad p \in (2, \infty).$$

These estimates imply inequalities (3.26). This completes the proof of the theorem.  $\square$

*Proof of Corollary 1.9.* From conditions (1.2) and (1.6) it follows that all auxiliary functions from Lemma 1.6 are well-defined and, in addition, condition (1.5) holds. From the properties of the FSS  $\{u, v\}$  of equation (1.7) (see Lemma 1.3) for all  $x \in \mathbb{R}$ , we obtain the inequalities

$$r(x) v'(x) \geq \int_{-\infty}^x q(t) v(t) dt, \quad r(x) |u'(x)| \geq \int_x^{\infty} q(t) u(t) dt. \tag{3.36}$$

Below we consecutively use equation (1.9), estimates (3.36), formula (1.12), Lemma 1.6, representation (2.2), inequalities (2.8) and (2.5), equality (1.18) and the definition (1.20) of  $A$ :

$$\begin{aligned}
1 &= r(x)v'(x)u(x) - r(x)u'(x)v(x) \geq u(x) \int_{-\infty}^x q(t)v(t)dt + v(x) \int_x^{\infty} q(t)u(t)dt \\
&= \int_{-\infty}^{\infty} q(t)G(x,t)dt \geq \int_{x-d(x)}^{x+d(x)} q(t)G(x,t)dt \\
&= \int_{x-d(x)}^{x+d(x)} q(t)\sqrt{\rho(t)\rho(x)} \exp\left(-\frac{1}{2}\left|\int_x^t \frac{d\xi}{r(\xi)\rho(\xi)}\right|\right) dt \\
&\geq c^{-1}\rho(x) \exp\left(-\frac{1}{2}\int_{x-d(x)}^{x+d(x)} \frac{d\xi}{r(\xi)\rho(\xi)}\right) \int_{x-d(x)}^{x+d(x)} q(t)dt \\
&\geq c^{-1} \exp\left(-\int_{x-d(x)}^{x+d(x)} \frac{d\xi}{r(\xi)h(\xi)}\right) h(x)d(x) \left[\frac{1}{2d(x)} \int_{x-d(x)}^{x+d(x)} q(t)dt\right] \geq c^{-1}A(h(x)d(x)) \Rightarrow \\
&\quad h(x)d(x) \leq cA^{-1} < \infty, \quad x \in \mathbb{R} \Rightarrow B = \sup_{x \in \mathbb{R}} h(x)d(x) \leq cA^{-1} < \infty.
\end{aligned}$$

The assertion of the Corollary now follows from Theorem 1.8.  $\square$

*Proof of Corollary 1.10.* Necessity.

Suppose equation (1.1) is correctly solvable in  $L_p$ ,  $p \in (1, \infty)$ . Since  $r \equiv 1$ , equalities (1.4) hold, and by Theorem 1.1 relations (1.5) and (1.6) are satisfied. Then all auxiliary functions from Lemma 1.6 and the function  $\tilde{d}(x)$  from Lemma 2.6 are well-defined (see [3]). Furthermore, from (1.18), (2.5) and (2.8), we get

$$\begin{cases} 1 = \int_{x-d(x)}^{x+d(x)} \frac{dt}{h(t)} \geq \frac{1}{2} \int_{x-d(x)}^{x+d(x)} \frac{dt}{\rho(t)} \geq \frac{1}{2e^2} \frac{d(x)}{\rho(x)}, & x \in \mathbb{R} \\ 1 = \int_{x-d(x)}^{x+d(x)} \frac{dt}{h(t)} \leq 2 \int_{x-d(x)}^{x+d(x)} \frac{dt}{\rho(t)} \leq 2e^2 \frac{d(x)}{\rho(x)}, & x \in \mathbb{R} \end{cases} \Rightarrow$$

$$2^{-1}e^{-2}d(x) \leq \rho(x) \leq 2e^2d(x), \quad x \in \mathbb{R}. \quad (3.37)$$

From (3.37), (2.5) and (2.7), it is easy to obtain the estimates

$$\begin{aligned} c^{-1}\tilde{d}(x) &\leq h(x), d(x) \leq c\tilde{d}(x), \quad x \in \mathbb{R} \Rightarrow \\ c^{-1}\tilde{d}(x)^2 &\leq h(x)d(x) \leq c\tilde{d}(x)^2, \quad x \in \mathbb{R} \end{aligned} \quad (3.38)$$

Since  $B < \infty$  in view of Theorem 1.8, according to (3.38), we have  $d_0 = \sup_{x \in \mathbb{R}} \tilde{d}(x) < \infty$ . We now obtain from (2.6):

$$2 = \tilde{d}(x) \int_{x-\tilde{d}(x)}^{x+\tilde{d}(x)} q(t)dt \leq d_0 \int_{x-d_0}^{x+d_0} q(t)dt \quad \Rightarrow \quad \inf_{x \in \mathbb{R}} \int_{x-d_0}^{x+d_0} q(t)dt \geq \frac{2}{d_0} > 0.$$

*Proof of Theorem 1.4. Sufficiency.*

Suppose (1.21) holds. Then it is clear that relations (1.5) and (1.6) are satisfied, and therefore all auxiliary functions from Lemma 1.6 and the function  $\tilde{d}$  from Lemma 2.6 are well-defined. Hence inequalities (3.38) remain true. From (1.21) it follows that there exists  $d_0 \gg 1$  such that

$$\int_{x-d_0}^{x+d_0} q(t)dt > \frac{2}{d_0}, \quad x \in \mathbb{R}.$$

Then  $\tilde{d}(x) \leq d_0$  for every  $x \in \mathbb{R}$ . Indeed, if for some  $x \in \mathbb{R}$  this inequality does not hold, then we have a contradiction:

$$2 = \tilde{d}(x) \int_{x-\tilde{d}(x)}^{x+\tilde{d}(x)} q(t)dt \geq d_0 \int_{x-d_0}^{x+d_0} q(t)dt > 2.$$

Thus  $\tilde{d}(x) \leq d_0$  for every  $x \in \mathbb{R}$ . But then from (3.38) we obtain  $B \leq cd_0^2 < \infty$ . It remains to refer to Theorem 1.8.  $\square$

#### 4. ADDITIONAL ASSERTIONS AND EXAMPLES

Below we present several applications of the results obtained above.

**Theorem 4.1.** [13] *Suppose conditions (1.2) hold, and, in addition,  $q_0 > 0$ , where*

$$q_0 = \inf_{x \in \mathbb{R}} q(x). \quad (4.1)$$

*Then equation (1.1) is correctly solvable in  $L_p$  for all  $p \in (1, \infty)$ .*

*Proof.* Since  $q_0 > 0$ , condition (1.6) holds, and by Lemma 1.6 the function  $d$  is well-defined. Then (see (1.20)):

$$A = \inf_{x \in \mathbb{R}} \frac{1}{2d(x)} \int_{x-d(x)}^{x+d(x)} q(t)dt \geq \inf_{x \in \mathbb{R}} \frac{1}{2d(x)} \int_{x-d(x)}^{x+d(x)} q_0 dt = q_0 > 0.$$

The statement of the theorem now follows from Corollary 1.9 and Remark 1.11.  $\square$

*Remark 4.2.* The proof of Theorem 4.1 given here was proposed in [3].

Thus, the requirement  $q_0 > 0$  to the function  $q$  is so strong that the answer to the question on I)–II) does not depend on the behavior of the function  $r$  (assuming (1.2)). In this connection, consider the opposite direction, i.e., find requirements to the function  $r$  under which the solution of the problem I)–II) does not depend on the behavior of the function  $q$  (in a certain framework).

**Theorem 4.3.** *Suppose conditions (1.2) hold, and, in addition,*

$$\int_{-\infty}^0 q(t)dt = \int_0^{\infty} q(t)dt = \infty. \quad (4.2)$$

*Then equation (1.1) is correctly solvable in  $L_p$  for all  $p \in (1, \infty)$  if  $\theta < \infty$ . Here*

$$\theta = \sup_{x \in \mathbb{R}} |x| \left( \int_{-\infty}^x \frac{dt}{r(t)} \right) \left( \int_x^{\infty} \frac{dt}{r(t)} \right). \quad (4.3)$$

*Proof.* From (1.2) and (4.2) it follows that relations (1.5) and (1.6) hold, and this means that the assumptions of Lemmas 1.3 and 1.6 and Theorem 1.8 are satisfied. Furthermore, from Lemma 1.3 it is easy to obtain the relations

$$\begin{aligned} u(x) &= v(x) \int_x^{\infty} \frac{dt}{r(t)v^2(t)}, \quad v(x) = u(x) \int_{-\infty}^x \frac{dt}{r(t)u^2(t)}, \quad x \in \mathbb{R} \Rightarrow \\ \rho(x) &= u(x)v(x) = v^2(x) \int_x^{\infty} \frac{dt}{r(t)v^2(t)} = u^2(x) \int_{-\infty}^x \frac{dt}{r(t)u^2(t)}, \quad x \in \mathbb{R} \end{aligned} \quad (4.4)$$

Note that  $\frac{1}{r} \in L_1$  because  $\theta < \infty$ . Therefore from (4.4) and Lemma 1.3, it is easy to obtain the following conceptual estimates:

$$\begin{aligned} \rho(x) &\leq u^2(x) \int_{-\infty}^x \frac{dt}{r(t)u^2(x)} = \int_{-\infty}^x \frac{dt}{r(t)}, \quad \text{if } x \leq 0 \\ \rho(x) &\leq v^2(x) \int_x^{\infty} \frac{dt}{r(t)v^2(x)} = \int_x^{\infty} \frac{dt}{r(t)}, \quad \text{if } x \geq 0. \end{aligned} \quad (4.5)$$

We shall also need the following simple consequence of Lemma 1.6:

$$d(x) \leq |x| \quad \text{for all } |x| \gg 1. \quad (4.6)$$

Below we consecutively use Lemma 2.5, (4.6), (4.5) and the condition

$$h(x)d(x) \leq 2\rho(x)x \leq 2x \int_x^{\infty} \frac{dt}{r(t)} \leq c < \infty \quad \text{for } x \gg 1; \quad (4.7)$$

$$h(x)d(x) \leq 2\rho(x)|x| \leq 2|x| \int_{-\infty}^x \frac{dt}{r(t)} \leq c < \infty \quad \text{for } x \ll -1. \quad (4.8)$$

Moreover, since the function  $\rho(x)d(x)$  is continuous and positive for  $x \in \mathbb{R}$  (see Lemma 1.6), the inequalities

$$0 < h(x)d(x) \leq 2\rho(x)d(x), \quad x \in \mathbb{R}$$

imply that the function  $h(x)d(x)$  is bounded on every finite interval. Together with (4.7) and (4.8), this give  $B < \infty$  (see (1.19)). It remains to apply Theorem 1.8.  $\square$

The following applications of Theorem 4.3 to concrete equations are subdivided into pairs so that it is interesting to compare the examples constituting each pair.

**Example 4.4.** Consider equation (1.1) with coefficients

$$r(x) = \begin{cases} 1 & \text{if } |x| \leq 1 \\ x^2 & \text{if } |x| \geq 1 \end{cases}, \quad q(x) = 1 + \cos |x|^\alpha, \quad x \in \mathbb{R}, \quad \alpha \in (0, \infty)$$

In this case the hypothesis of Theorem 4.3 are obviously satisfied,  $\theta < \infty$  and therefore such an equation is correctly solvable in  $L_p$  for all  $p \in (1, \infty)$  regardless of  $\alpha \in (0, \infty)$ .

**Example 4.5.** Consider equation (1.1) with coefficients

$$r(x) \equiv 1, \quad x \in \mathbb{R}; \quad q(x) = 1 + \cos |x|^\alpha, \quad \alpha \in (0, \infty), \quad x \in \mathbb{R}.$$

Then (see [5]) such an equation is correctly solvable in  $L_p$  if and only if  $\alpha \geq 1$ .

*Remark 4.6.* The statement of Example 4.5 becomes completely obvious if one compares the criterion for correct solvability  $m(a) > 0$ ,  $a \in (0, \infty)$  (see (1.21)) with the different behavior at infinity of the graphs of  $q$  for  $\alpha \in (0, 1)$  and  $\alpha \in [1, \infty)$  in the zeros of the function  $q$ .

Let us now consider two examples of a different type.

**Example 4.7.** Consider equation (1.1) with coefficients

$$r(x) = \begin{cases} 1, & \text{if } |x| \leq 1 \\ x^2, & \text{if } |x| \geq 1 \end{cases}, \quad q(x) = \begin{cases} 1, & \text{if } |x| \leq 1 \\ \frac{1}{\sqrt{|x|}}, & \text{if } |x| \geq 1 \end{cases} \quad (4.9)$$

In the case (4.9) all the hypotheses of Theorem 4.3 are obviously satisfied,  $\theta < \infty$  and therefore such an equation is correctly unsolvable in  $L_p$  for all  $p \in (1, \infty)$ .

**Example 4.8.** Consider equation (1.1) with coefficients  $r \equiv 1$ ,  $x \in \mathbb{R}$  and  $q$  satisfying condition (1.2). If, in addition,

$$\lim_{x \rightarrow -\infty} q(x) = 0 \quad \text{or} \quad \lim_{x \rightarrow \infty} q(x) = 0, \quad (4.10)$$

equation (1.1) is correctly unsolvable in  $L_p$  for all  $p \in (1, \infty)$  (see Corollary 1.10).

Let us now consider the equation of direct application of Theorem 1.8 to particular equations (1.1). Since it is usually impossible to find an explicit form for the functions  $h$  and  $d$ ,

a general method of applying the theorem consists of obtaining two-sided, sharp by order estimates for the functions  $h$  and  $d$  at infinity. Clearly, such inequalities lead to exact estimates of the function  $h \cdot d$  at infinity and then to a complete answer to the question on the finiteness of  $B$  (see the proof of Theorem 4.3) and, finally, to a concluding statement using Theorem 1.8.

Below we present an example where this scheme is realized for the case of equation (1.1) with coefficients (4.9). To this end, we present the following assertion.

**Lemma 4.9.** *Suppose conditions (1.2) and (1.6) hold. For a given  $x \in \mathbb{R}$  let us introduce the functions  $F_1(\eta)$ ,  $F_2(\eta)$  and  $F_3(\eta)$  with  $\eta \geq 0$ :*

$$F_1(\eta) = \int_{x-\eta}^x \frac{dt}{r(t)} \cdot \int_{x-\eta}^x q(t)dt, \quad F_2(\eta) = \int_x^{x+\eta} \frac{dt}{r(t)} \cdot \int_x^{x+\eta} q(t)dt,$$

$$F_3(\eta) = \int_{x-\eta}^{x+\eta} \frac{dt}{r(t)h(t)}.$$

*Then the following assertions hold (see Lemma 1.6):*

- a) *the inequality  $\eta \geq d_1(x)$  ( $0 \leq \eta \leq d_1(x)$ ) holds if and only if  $F_1(\eta) \geq 1$  ( $F_1(\eta) \leq 1$ );*
- b) *the inequality  $\eta \geq d_2(x)$  ( $0 \leq \eta \leq d_2(x)$ ) holds if and only if  $F_2(\eta) \geq 1$  ( $F_2(\eta) \leq 1$ );*
- c) *the inequality  $\eta \geq d(x)$  ( $0 \leq \eta \leq d(x)$ ) holds if and only if  $F_3(\eta) \geq 1$  ( $F_3(\eta) \leq 1$ ).*

*Proof of Lemma 4.9. Necessity.*

All assertions of the Lemma are proved in the same way. Consider, say, b). Clearly,

$$F_2'(\eta) = \frac{1}{r(x+\eta)} \int_x^{x+\eta} q(t)dt + q(x+\eta) \int_x^{x+\eta} \frac{dt}{r(t)}, \quad (4.11)$$

and therefore  $F_2'(\eta) \geq 0$  in view of (1.2). Then if  $\eta \geq d_2(x)$ , then  $F_2(\eta) \geq F_2(d_2(x)) = 1$ .

*Proof of Lemma 4.9. Sufficiency.* Let  $F_2(\eta) \geq 1$ . Assume the contrary:  $\eta < d_2(x)$ . Since  $F_2(\eta) \geq 1$ , we have

$$\int_x^{x+\eta} q(t)dt > 0 \quad \Rightarrow \quad F_2'(\eta) > 0 \quad (\text{see (4.11)}) \quad \Rightarrow \quad 1 \leq F_2(\eta) < F_2(d_2(x)) = 1,$$

contradiction. Hence  $\eta \geq d_2(x)$ . □

*Remark 4.10.* Lemma 4.9 is an efficient tool for proving estimates of the functions  $h$  and  $d$  (see below). In particular, in [4] it was used for a meaningful class of equations (1.1) in order to get a priori, sharp by order, two-sided estimates of these functions expressed in terms of the functions  $r$  and  $q$ . See [3, 4] for a detailed exposition of proofs and applications of such inequalities. In the case (4.9), a priori estimates from [4] are not applicable because of the fast growth of the function  $r/q$ .

Let us introduce the following notation. Let  $\alpha(x)$  and  $\beta(x)$  be continuous positive functions for  $x \in (a, b)$  ( $-\infty \leq a < b \leq \infty$ ). We write  $\alpha(x) \asymp \beta(x)$  for  $x \in (a, b)$  if there exists a constant  $c \in [1, \infty)$  such that

$$c^{-1}\alpha(x) \leq \beta(x) \leq c\alpha(x) \quad \text{for all } x \in (a, b).$$

**Lemma 4.11.** *In the case (4.9), the following relations hold (see Lemma 1.6):*

$$\varphi(x) \asymp \begin{cases} \frac{1}{|x|} & \text{if } x \ll -1 \\ \frac{1}{\sqrt{|x|}} & \text{if } x \gg 1 \end{cases} \quad \psi(x) \asymp \begin{cases} \frac{1}{\sqrt{|x|}} & \text{if } x \ll -1 \\ \frac{1}{|x|} & \text{if } x \gg 1 \end{cases} \quad (4.12)$$

$$h(x) \asymp \frac{1}{|x|} \quad \text{if } |x| \gg 1. \quad (4.13)$$

*Proof.* Both relations of (4.12) are proved in the same way. Let us check, say, the second one. It is easy to see that in the case (4.9) the following equality holds (see Lemma 4.9):

$$F_2(\eta) = \frac{2\eta^2}{x(x+\eta)(\sqrt{|x|} + \sqrt{|x+\eta|})}, \quad \text{if } [x, x+\eta] \cap [-1, 1] = \emptyset. \quad (4.14)$$

Let  $x \gg 1$ . Then in view of (4.14) we have the inequalities

$$F_2(\eta) \big|_{\eta=4x^2} > 1, \quad F_2(\eta) \big|_{\eta=\frac{x^2}{4}} < 1,$$

and therefore by Lemma 4.9, we obtain

$$4^{-1}x^2 \leq d_2(x) \leq 4x^2 \quad \text{for } x \gg 1.$$

Similarly, for  $x \ll -1$  in view of (4.14), we have the inequalities:

$$F_2(\eta) \big|_{\eta=|x|-\sqrt{|x|}} > 1, \quad F_2(\eta) \big|_{\eta=|x|-4\sqrt{|x|}} < 1,$$

and therefore by Lemma 4.9 we obtain:

$$|x| - 4\sqrt{|x|} \leq d_2(x) \leq |x| - \sqrt{|x|} \quad \text{for } x \ll -1.$$

Thus,

$$d_2(x) \asymp \begin{cases} x^2 & \text{if } x \gg 1 \\ |x| & \text{if } x \ll -1 \end{cases} \quad (4.15)$$

Relations (4.12) for  $\psi$  follows from (4.15), the definition of the function  $\psi$  and (4.9). Formula (4.13) follows from (4.12) and the definition of the function  $h$ .  $\square$

*Remark 4.12.* If one follows the main method of Lemma 4.11, then using (4.13) and Lemma 4.9 one can obtain two-sided, sharp by order estimates for the function  $d$ . However, it is worth noting that in this situation, as in many others, one can always “economize” technical work if the already obtained estimate for the function  $h$  shows that  $h(x)|x| \leq c < \infty$  for all  $|x| \gg 1$ .

Indeed, in the latter case the proof of the estimates for the function  $d$  becomes superfluous because one can replace the sharp by order upper estimate of the function  $d$  with the rougher a priori estimate (4.6) without changing the results on  $B$  (see Theorem 1.8).

Let us now show, say, that  $B < \infty$  (see (1.19)). From (4.13) and (4.6), we get the inequalities

$$h(x)d(x) \leq \frac{c}{|x|} \cdot |x| = c < \infty \quad \text{for all} \quad |x| \gg 1. \quad (4.16)$$

Since on every finite segment  $[a, b]$  the function  $h(x)d(x)$ ,  $x \in [a, b]$  is bounded, from (4.16) we conclude that indeed  $B < \infty$ . By Theorem 1.8, this implies that in the case (4.9), equation (1.1) is correctly solvable in  $L_p$  for all  $p \in (1, \infty)$ .

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DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, BEN-GURION UNIVERSITY OF THE NEGEV,  
P.O.B. 653, BEER-SHEVA, 84105, ISRAEL

DEPARTMENT OF MATHEMATICS, BAR-ILAN UNIVERSITY, 52900 RAMAT GAN, ISRAEL